Quasielliptical motion of an electron in an electric dipole field

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A reformulation of a known conservation law is employed to study certain zero energy trajectories of a classical point test charge (e.g., a classical electron) moving under the influence of the electrostatic force due to a fixed electric dipole. It is found that the motion takes place along a ''folded'' ellipse with foci at the dipole charges and a fold along an axis perpendicular to the line joining these foci and lying in the plane of the motion. The motion is determined to be periodic with period $T(d,y_0) = \sqrt{m}\sqrt{4\pi\epsilon_0/qQ}(\sqrt{\pi/2}[\Gamma(1/4)/\Gamma(3/4)])[\frac{1}{3}d^{3/2}+(2/\sqrt{d})y_0^2]$, where d is the separation of the dipole charges that are placed symmetrically about the origin on the x axis, and y_0 is the initial y position of the test charge that starts from rest at $(0,y_0)$. [S1063-651X(99)01204-0]

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I. INTRODUCTION

Both the motion of a test charge trapped in a static electric dipole field and its gravitational cousin, the problem of ''two centers of gravitation,'' have a long history [1–5] and have been formally treated in many treatises on advanced mechanics [6–9]. Over 200 years ago Euler proved the integrability of the gravitational problem. However, few physical conclusions have been drawn from the mathematical results, which are formal relations between elliptic integrals. The complexity of these integrals and their inverses has to date precluded a physically meaningful interpretation of the motion, especially for the general case of bound motion. The difficulty in translation from mathematics to physics is largely due to the fact that the test-charge motion for bound motion is extremely complex, perhaps even chaotic. This is unfortunate, since this simple physical system is of general interest and is studied at some level by many physicists.

It is also well known that the Hamilton-Jacobi equation for this problem is separable in an elliptical coordinate system [7–9]. For this problem the constants of separation were identified a very long time ago. A separation constant associated with the Hamilton-Jacobi equation provides a formal definition of a first integral. However, a simple physical interpretation of one of the separation constants has never been given. In practical terms this has meant that little progress has been made in physically understanding the test-charge motion since the original work of Euler. In this paper a presumably new physical interpretation of this separation constant is given in Eq. (3), which provides, almost immediately, an understanding of the zero energy motion that starts from rest at $(0,y_0)$.

We shall restrict our discussion to the special case of motion in a plane, which we take to be the x-y plane. The total mechanical energy E is conserved in this system, while angular momentum is not. Instead the separation constant of Eq. (3) provides a second conserved quantity. The $E>0$ motion is unbounded; the test charge escapes to infinity. The $E<0$ motion is complex and not the focus here, although we hope this work describes a starting point for investigating this possibly chaotic motion of the test charge. Zero energy motion that starts from rest at $(0,y_0)$ is quantifiable and a natural starting point for more general investigations. Let us now turn to the formulation of this problem.

II. ZERO ENERGY DYNAMICS

Let $2a$ be the separation of electric dipole charges $\{q_1=-q,q_2=q>0\}$ that are placed symmetrically about the origin on the x axis at points $(-a,0)$ and $(a,0)$, respectively, in a Cartesian coordinate system. A test charge of mass m and electric charge $-Q<0$ moves under the influence of electrostatic forces due to this fixed electric dipole. If the test charge is located at (x,y), then the potential energy of this system is $U(x,y)=qQ/4\pi\epsilon_0\{1/r_1-1/r_2\}$, where r_i is the distance from the test charge to the ith fixed charge, $r_{1,2}^2=(x\pm a)^2+y^2$.

In terms of elliptic coordinates $(\zeta,\theta), x=a\cosh(\zeta)\cos(\theta)$ and $y=a\sinh(\zeta)\sin(\theta)$, so that $r_1+r_2/2=a\cosh(\zeta)$ and $r_1-r_2/2=a\cos(\theta)$. In elliptic coordinates the potential energy of the system is

$$\begin{aligned}V(\zeta,\theta)&=U(x(\zeta,\theta),y(\zeta,\theta))\\&=-(qQ/\pi\epsilon_0 a)\cos(\theta)/\cosh(2\zeta)-\cos(2\theta)\\&=-V_0[\cos(\theta)/\cosh(2\zeta)-\cos(2\theta)],\end{aligned}$$

where $V_0=qQ/\pi\epsilon_0 a$.

Let the dot denote differentiation with respect to $t,\dot{f}=df/dt$. In terms of elliptic coordinates the kinetic energy is $T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)=\frac{1}{4}ma^2[\cosh(2\zeta)-\cos(2\theta)](\dot{\zeta}^2+\dot{\theta}^2)$. The Lagrangian is $L=T-V$ and the canonical momenta are $p_\zeta=\partial L/\partial\dot{\zeta}=\frac{1}{2}ma^2[\cosh(2\zeta)-\cos(2\theta)]\dot{\zeta}$ and $p_\theta=\partial L/\partial\dot{\theta}=\frac{1}{2}ma^2[\cosh(2\zeta)-\cos(2\theta)]\dot{\theta}$.

The Hamiltonian of the system is $H=p_i\dot{q}^i-L=p_\zeta^2+p_\theta^2/ma^2[\cosh(2\zeta)-\cos(2\theta)]-V_0[\cos(\theta)/\cosh(2\zeta)-\cos(2\theta)]=p_\zeta^2+p_\theta^2-ma^2V_0\cos(\theta)/ma^2[\cosh(2\zeta)-\cos(2\theta)]$ which is conserved. Let E denote the constant value of H on a given

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trajectory. The Hamilton-Jacobi equation $H(\zeta,\theta,\partial S/\partial\zeta,\partial S/\partial\theta)+\partial S/\partial t=0$ is separable, $S=-Et+S_1(\zeta)+S_2(\theta)$. $S_1(\zeta)$ and $S_2(\theta)$ solve $(\partial S_1/\partial\zeta)^2+(\partial S_2/\partial\theta)^2=ma^2V_0\cos(\theta)+ma^2[\cosh(2\zeta)-\cos(2\theta)]E = ma^2V_0\cos(\theta)+2ma^2[\cosh^2(\zeta)-\cos^2(\theta)]E$. Separating variables yields

$$\left(\frac{\partial S_1}{\partial\zeta}\right)^2-2ma^2E\cosh^2(\zeta)=\text{const}\equiv\alpha \tag{1}$$

and

$$-\left(\frac{\partial S_2}{\partial\theta}\right)^2+ma^2[V_0\cos(\theta)-2E\cos^2(\theta)]=\alpha. \tag{2}$$

α and E characterize the classes of the different possible trajectories. Substituting $\partial S/\partial\zeta=\partial S_1/\partial\zeta=p_\zeta=\partial L/\partial\dot{\zeta}$ into Eq. (1) yields (after some algebra) an alternative evaluation for the separation constant α as

$$\alpha=-2ma^2E\cosh^2(\zeta)+m^2a^4[\cosh^2(\zeta)-\cos^2(\theta)]^2\dot{\zeta}^2$$
$$=-m\frac{E(r_1+r_2)^2}{2}+m^2\frac{r_1^2r_2^2}{(r_1+r_2)^2-4a^2}(\dot{r}_1+\dot{r}_2)^2. \tag{3}$$

Surprisingly, this physical interpretation of the separation constant seems to be new, and has made the analysis of the zero energy motion possible (below).

Similarly, substituting $\partial S/\partial\theta=\partial S_2/\partial\theta=p_\theta=\partial L/\partial\dot{\theta}$ into Eq. (2) yields

$$\alpha=-m^2a^4[\cosh^2(\zeta)-\cos^2(\theta)]^2\dot{\theta}^2+ma^2V_0\cos(\theta)$$
$$-2ma^2E\cos^2(\theta). \tag{4}$$

Eliminating dt from these two equations gives the equation for $\zeta=\zeta(\theta)$:

$$\left(\frac{d\zeta}{d\theta}\right)^2=\frac{\alpha+2ma^2E\cosh^2(\zeta)}{ma^2V_0\cos(\theta)-2ma^2E\cos^2(\theta)-\alpha}. \tag{5}$$

A complete integral of the Hamilton-Jacobi equation is provided by [7]

$$S_1(\zeta)=\pm\int_{\theta=\text{const}}\sqrt{\alpha+2ma^2E\cosh^2(\zeta)}\;d\zeta,$$

$$S_2(\theta)=\pm\int_{\zeta=\text{const}}\sqrt{-\alpha+ma^2V_0\cos(\theta)-2ma^2E\cos^2(\theta)}$$
$$\times d\theta. \tag{6}$$

Let us consider the $E=0$ motion that starts from rest on the y axis at $(0,y_0)$. Since the test charge starts from rest, $\dot{r}_1+\dot{r}_2=0$ initially. From Eq. (3) we see that $\alpha=0$ for this trajectory. But the pair $(E,\alpha)=(0,0)$ classifies this motion, each parameter retaining its value throughout the course of the motion. We conclude that $\dot{r}_1+\dot{r}_2\equiv 0\Rightarrow r_1+r_2=\text{const}=2\sqrt{a^2+y_0^2}$, which describes a (folded) ellipse with foci at the dipole charge positions.

Typically, the test-charge motion proceeds from $(0,y_0>0)$ to the right and downward along an arc that bends around the fixed positive dipole charge, always to the right of $(a,0)$, intersecting the x axis at $\sqrt{a^2+y_0^2}$. The motion continues along a symmetrical (reflected) arc to $(0,-y_0)$ where the test charge is instantaneously at rest (by energy conservation). The motion then reverses and finally arrives back at the starting point at $(0,y_0)$ after an elapsed time T. The orbit follows an ellipse folded over onto itself, the y axis coinciding with the fold axis. Before folding along the y axis, the ellipse foci coincide with the locations of the fixed charges of the electric dipole. This type of $E=0$ motion is periodic.

A general expression for the period may be found as follows. For virtual paths conserving energy and whose variations have fixed coordinate end points, $\delta S+E\,\delta t=0$, where $S=\int_{t_0}^{t}L\,dt=\int_{t_0}^{t}(p_i\dot{q}^i-H)dt=\int_{t_0}^{t}p_i\dot{q}^i dt-E(t-t_0)=S_0-E(t-t_0)$. Here $S_0=\int_{t_0}^{t}p_i\dot{q}^i dt=S_1+S_2$ is Hamilton's characteristic function, also called the *abbreviated action*, and t is the time at which a point on the trajectory is occupied [7,8]. The variation of S with respect to E is $\delta S=\delta S_0-E\,\delta t-(t-t_0)\,\delta E$. Therefore, as is very well known, $t+\text{const}=\partial S_0/\partial E$ [7,8]. For motion about a folded ellipse $r_1+r_2=2a\cosh(\zeta)=2\sqrt{a^2+y_0^2}=\text{const}$, and the period of the zero energy motion is therefore given by

$$T(a,y_0)=4\left(\left[\frac{\partial S_0}{\partial E}\right]_{(\zeta,\theta=\pi/2)}-\left[\frac{\partial S_0}{\partial E}\right]_{(\zeta,\theta=0)}\right), \tag{7}$$

where E and α are set equal to zero after performing the differentiation. What should be differentiated? One knows that the complete integral to the Hamilton-Jacobi equation contains $n+1$ constants, one of which is purely additive, where n is the number of coordinate degrees of freedom. In this case $n=2$ and the constants are E and α, these parameters classifying the different trajectory classes. One also knows that Hamilton's characteristic function defines a canonical transformation from ζ to a constant canonical coordinate α. However, the previous variation of S is with respect to E [and t, with $\delta t=(\partial t/\partial E)\,\delta E$, where $\delta S+E\,\delta t=0$] with the coordinate end points held fixed (Maupertuis' principle). The independent parameters in this variation are accordingly E and the original coordinates (ζ,θ), since otherwise there is no way to ensure that $\delta\zeta=0$ and $\delta\theta=0$ at the end points. Thus, hidden in this variation is an implicit canonical transformation from α back to the original coordinates. Hence $\alpha=\alpha(E,\zeta,p_\zeta)$ during this variation, as manifested by Eq. (1). It should be emphasized that in general $\partial\alpha/\partial E\neq 0$ under this variation.

For purposes of calculating $\partial\alpha/\partial E$, only geometrical (kinematical) but not dynamical substitutions are allowed in Eq. (3). For example, we do not substitute in this identity for, say, $\dot{\zeta}$ by solving for $\dot{\zeta}$ in the expression for the conserved total energy E. Only an explicit E dependence contributes to the partial derivative. Implicit contributions such as $(\partial\alpha/\partial\dot{\zeta})(\partial\dot{\zeta}/\partial E)$ are not included. To evaluate this partial derivative, we use Eq. (3), which gives $\partial\alpha/\partial E=-2ma^2\cosh^2(\zeta)=-m[(r_1+r_2)^2/2]=-2m(a^2+y_0^2)$. Hence $\partial S_1/\partial E=0$ (the numerator of this integral vanishes). Therefore $T(a,y_0)=4\Delta[\partial S_0/\partial E]=4\Delta[\partial S_2/\partial E]=2\int_0^{\pi/2}[-(\partial\alpha/\partial E)-2ma^2\cos^2(\theta)/$

$\sqrt{ma^2V_0\cos(\theta)}]d\theta = 4\sqrt{\pi\epsilon_0/qQ}\ \sqrt{m/a}\ \int_0^{\pi/2}[a^2\sin^2(\theta)+y_0^2/\sqrt{\cos(\theta)}]d\theta$. Since $\int_0^{\pi/2}[a^2\sin^2(\theta)+y_0^2/\sqrt{\cos(\theta)}]d\theta=\sqrt{\pi}[\Gamma(1/4)/\Gamma(3/4)](\frac{1}{3}a^2+\frac{1}{2}y_0^2)$, one finds that

$$T(a,y_0)=\sqrt{m}\sqrt{\frac{4\pi\epsilon_0}{qQ}}\left(\sqrt{\pi}\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)\left(\frac{2}{3}a^{3/2}+\frac{1}{\sqrt{a}}y_0^2\right)$$

$$=\sqrt{m}\sqrt{\frac{4\pi\epsilon_0}{qQ}}\left(\sqrt{\frac{\pi}{2}}\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)\left(\frac{1}{3}d^{3/2}+\frac{2}{\sqrt{d}}y_0^2\right), \tag{8}$$

where $d=2a$ is the separation of the dipole charges. Note that $\sqrt{\pi/2}[\Gamma(1/4)/\Gamma(3/4)]=K(1/\sqrt{2})$, where K is the complete elliptic integral of the first kind.

A simpler but less instructive approach to calculating the period is to solve Eq. (4) for dt,

$$dt=ma^2\frac{\cosh^2(\zeta)-\cos^2(\theta)}{\sqrt{ma^2V_0\cos(\theta)-2ma^2E\cos^2(\theta)-\alpha}}d\theta, \tag{9}$$

evaluate this at $E=0$ and $\alpha=0$, and compute four times the integral from $\theta=0$ to $\theta=\pi/2$. This gives exactly the same result as above, but does not shed light on the nature of the variations δS and $\delta\alpha$.

III. CONCLUSION

The $E=0$ motion in a static electric dipole field of a test charge that starts from rest on the symmetry axis perpendicular to the line joining the dipole charges has been investigated. It has been shown that the $(E=0,\alpha=0)$ motion is periodic and takes place on a folded ellipse with period given by Eq. (8). To obtain this result we have used the very well known fact that the Hamilton-Jacobi equation for this system is separable. An apparently new simple physical interpretation of the separation constant α in Eq. (3) has been found, which leads to a simple geometrical interpretation of the motion.

It is of interest to apply a similar geometrical analysis to the classical analog of the Born-Oppenheimer states of H_2^+ [10]. The potential energy in this case is $U_{H_2^+}(x,y)=-V_0\{1/r_1+1/r_2\}$ where $V_0=qQ/\pi\epsilon_0 a$, and $q=Q$ equals the proton charge. To perform the analysis we consider the transformation $r_1+r_2/2=a\cosh(\zeta)\equiv\sigma$ and $r_1-r_2/2=a\cos(\theta)\equiv\delta$, from which $r_1=\sigma+\delta$ and $r_2=\sigma-\delta$. Comparison with the electric dipole potential energy $U(x,y)=V_0\{1/r_1-1/r_2\}$ shows that if one interchanges σ and δ and makes the replacement $V_0\to -V_0$, then the electric dipole potential energy maps into $U_{H_2^+}$. After rewriting α in terms of σ and δ and then interchanging σ and δ, one finds that $\alpha\to m(E/2)(r_1-r_2)^2-m^2[r_1^2r_2^2(\dot r_2-\dot r_1)^2/(r_2-r_1)^2-4a^2]$. One concludes that for hyperbolic motion $(r_2-r_1=\text{const})$ $\alpha=\text{const}$. Also, if $E=0$, then $\alpha=0$ and the motion is along a folded degenerate hyperbola (viz., the y axis). Now, in the quantum mechanical treatment of the H_2^+ ion the existence of a conservation law associated to α leads to a "hidden" symmetry that allows electron terms of the same symmetry to cross; this intersection is otherwise forbidden by the Neumann-Wigner noncrossing rule. Classically, this conserved quantity is

$$m\frac{E}{2}(r_1-r_2)^2-m^2\frac{r_1^2r_2^2(\dot r_2-\dot r_1)^2}{(r_2-r_1)^2-4a^2}. \tag{10}$$

One important aspect of the *quantum* mechanical electron-dipole interaction concerns the existence of bound states, which can only exist if the electric dipole moment exceeds some critical value. We may hope to gain some insight into this phenomenon by employing the semiclassical Bohr-Sommerfeld quantization rule $\oint p\,dq=(n+\frac{1}{2})h$ to "quantize" this system. Consider $\oint p_\theta d\theta=(n+\frac{1}{2})h$. Solving for p_θ from Eq. (2) and substituting in this integral yields $2\int_{-\pi/2}^{\pi/2}\sqrt{ma^2[V_0\cos(\theta)-2E\cos^2(\theta)]-\alpha}\,d\theta=(n+\frac{1}{2})h$. Evaluating this for $\alpha=0$ and $E=0$ gives $8a\sqrt{mV_0}\text{Ei}(\pi/4|2)=(n+\frac{1}{2})h$, where $\text{Ei}(\phi|m)$ is the elliptic integral of the second kind and $\text{Ei}(\pi/4|2)\approx 2.396\,28$. Squaring for the case $n=0$ gives $64ma^2(qQ/\pi\epsilon_0 a)\text{Ei}(\pi/4|2)^2=h^2/4$, which implies that $qa=\pi\epsilon_0 h^2/256mQ\text{Ei}(\pi/4|2)^2$. We find that semiclassically there is also a threshold for bound states. However, there may be no purely classical analog to this minimum critical value.

A generalization of Eq. (8) may not exist for negative energy motion. Indeed, the $E<0$ motion appears to be chaotic. At present we have not found initial coordinates that produce periodic motion. Let the origin for polar coordinates $(u=1/r,\theta)$ be the position of the right charge of the dipole, where r is the distance of the test charge to the origin. Following a general procedure advocated by McCauley [11], let us define the sequence $\{\dot u_n\}$ as the value of $\dot u$ when $\dot\theta=0$ and $y>0$. We have found that there exists an iterated map $\dot u_{n+1}=f(\dot u_n)$, with $f(x)=\alpha-\beta^2/\alpha'+x$, that represents this series. α,β and α' are functions of initial position (the test charge always starts from rest). f is not bounded above or below. It has (in general, complex) fixed points $\alpha-\alpha'/2\pm\sqrt{(\alpha+\alpha'/2)^2-\beta^2}$. Iterations of this map yield a continued fraction representation of an irrational number. This is not a typical property of a chaotic map. In fact, all $E<0$ motion is bounded by "initial" equipotential curves, so perhaps exponential divergence of nearby trajectories is not a particularly useful characterization of chaotic motion here. The motion gets "scrambled" by a slingshot effect. There are occasions when the test charge approaches the attracting fixed charge of the dipole nearly head-on. Whether the test charge is whipped around the fixed charge in the counterclockwise or clockwise direction can depend on digits in initial coordinates arbitrarily far to the right of the decimal point. The complex $E<0$ motion of this charge has the potential to provide a rich laboratory for theoretical experimentation in nonlinear dynamics.

[1] L. Euler, Mem. Acad. Sci. Berlin **16**, 228 (1760).
[2] L. Euler, Nov. Commun. Acad. Sci. Petrop., **10**, 207 (1764).
[3] J.L. Lagrange, Mem. de l'Acad. de Paris, **IX**, 97 (1772).
[4] A.M. Hiltebeitel, Am. J. Math. **xxxii**, 337 (1911).
[5] X. Charlier, *Die Mechanik des Himmels* (W. de Gruyter, Berlin, 1927).
[6] E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge University Press, London, 1965).
[7] H.C. Corben and P. Stehle, *Classical Mechanics*, 2nd ed. (Dover, New York, 1994).
[8] L.D. Landau and E.M. Lifshitz, *Mechanics*, 2nd ed. (Addison-Wesley, Reading, 1969).
[9] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. (Springer-Verlag, New York, 1989).
[10] S.S. Gershtein and V.D. Krivchenkov, Zh. Éksp. Teor. Fiz. **40**, 692 (1960) [Sov. Phys. JETP **13**, 1044 (1961)].
[11] J.L. McCauley, *Chaos, Dynamics and Fractals* (Cambridge University Press, Chicago, 1994).